Conditions for C-α continuity of Bezier Curves

10th November 2001

Peter J. Taylor [1]

1 Abstract

It is proved that for C-α continuity of Bezier curves of degree n, the constraints required at the join between curves P and Q are:

$$\forall \beta \leq \alpha \,.\, Q_\beta = 2^\beta \sum_{k=0}^{\beta} (-2)^k \binom{\beta}{k} P_{n-k}$$

2 Proof

A Bezier curve, P of degree n is defined by $n+1$ control points, P_i and parameterised by t:

$$P(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_i \tag{1}$$

The coefficients are clearly Binomial coefficients and if we take the bold algebraic step of identifying P_i with ${x_P}^i$ for a dummy variable x_P, this becomes:

$$P(t) = (1 - t + tx_P)^n \tag{2}$$

We can differentiate with respect to t consistently in this Algebra, since:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum f_i(t){x_P}^i\right) = \sum \frac{\mathrm{d}}{\mathrm{d}t}\left(f_i(t){x_P}^i\right) = \sum \left(\frac{\mathrm{d}}{\mathrm{d}t} f_i(t)\right) {x_P}^i$$

Then it is easily shown by induction that:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} P(t) = n^{\underline{\alpha}} \, (x_P - 1)^\alpha (1 - t + tx_p)^{n-\alpha} \tag{3}$$

Since P is thus shown to be infinitely differentiable, the only constraints placed by C-α continuity are at the joins between Bezier curves. Suppose we have two

[1] The author would like to thank Andrew D. Rogers for typesetting this paper in LaTeX

curves of degree n: P and Q. Then for C-α continutity we require C-(α-1) continuity and:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}P(t)\bigg|_{t=1} = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}Q(t)\bigg|_{t=0} \tag{4}$$

By substitution using (3), this reduces to:

$$(x_P - 1)^\alpha {x_P}^{n-\alpha} = (x_Q - 1)^\alpha \tag{5}$$

Thus:

$${x_P}^{n-\alpha}\sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i {x_P}^{\alpha-i} = \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^{\alpha-j}{x_Q}^j \tag{6}$$

Or:

$$\sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i P_{n-i} = (-1)^\alpha\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j Q_j \tag{7}$$

It is easy to show by induction on α that this condition, combined with the constraints for C-(α-1) continuity, allows us to express Q_j in the form:

$$Q_j = \sum_{k=0}^{j} P_{n-k}T_{j,k} \tag{8}$$

Where $T_{a,b}$ is to be found.

Evaluation of the first few terms and a lookup in the On-Line Encyclopedia Of Integer Sequences [1] turned up sequence A038207 which appeared very similar. From this I conjectured:

$$T_{j.k} = (-1)^k 2^{j-k}\binom{j}{k} \tag{9}$$

The rest of this paper is a proof that this satisfies equation (7).

Substituting (8) into (7) gives:

$$\sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i P_{n-i} = (-1)^\alpha\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\sum_{k=0}^{j}P_{n-k}T_{j,k} \tag{10}$$

Whence:

$$(-1)^\alpha\sum_{i=0}^{\alpha}\binom{\alpha}{i}(-1)^i P_{n-i} = \sum_{j=0}^{\alpha}\sum_{k=0}^{j}\binom{\alpha}{j}(-1)^j T_{j,k}P_{n-k} \tag{11}$$

If we consider the coefficient of P_{n-r}, where $r \leq \alpha$, we find

$$(-1)^\alpha \binom{\alpha}{r} (-1)^r = \sum_{j=r}^{\alpha} \binom{\alpha}{j} (-1)^j T_{j,r} \tag{12}$$

So my task is complete if I can prove:

$$\sum_{j=r}^{\alpha} \binom{\alpha}{j} (-1)^j (-1)^r 2^{j-r} \binom{j}{r} \stackrel{?}{=} (-1)^{\alpha+r} \binom{\alpha}{r} \tag{13}$$

The left-hand side can be tidied up and the 2^r transferred to give:

$$\sum_{j=r}^{\alpha} (-2)^j \binom{\alpha}{j} \binom{j}{r} \stackrel{?}{=} (-1)^\alpha 2^r \binom{\alpha}{r} \tag{14}$$

Let:

$$S(\alpha, j) \stackrel{\text{def}}{=} (-2)^j \binom{\alpha}{j} \binom{j}{r} \tag{15}$$

The the Maple package EKHAD [2] by D. Zeilberger tells you that if:

$$G(\alpha, j) \stackrel{\text{def}}{=} \frac{(r-j)(\alpha+1)}{\alpha-j+1} S(\alpha, j) \tag{16}$$

Then:

$$(\alpha+1) S(\alpha, j) + (\alpha - r + 1) S(\alpha+1, j) = G(\alpha, j+1) - G(\alpha, j) \tag{17}$$

This is easily verified by hand. Then summing over all j,

$$(\alpha+1) \sum_j S(\alpha, j) + (\alpha - r + 1) \sum_j S(\alpha+1, j) = 0 \tag{18}$$

since G has compact support ($S(\alpha, j) = 0$ unless $0 \leq j \leq \alpha$). Therefore:

$$(\alpha+1) \sum_j (-2)^j \binom{\alpha}{j} \binom{j}{r} = (r - \alpha - 1) \sum_j (-2)^j \binom{\alpha+1}{j} \binom{j}{r} \tag{19}$$

But since $\binom{a}{b}$ has compact support, we can restrict j thus:

$$(\alpha+1) \sum_{j=r}^{\alpha} (-2)^j \binom{\alpha}{j} \binom{j}{r} = (r - \alpha - 1) \sum_{j=r}^{\alpha+1} (-2)^j \binom{\alpha+1}{j} \binom{j}{r} \tag{20}$$

I am now ready to prove (14) by induction on α:

Case $\alpha = 0$:

$$\begin{aligned} \text{LHS of (14)} &= 0 \\ \text{RHS of (14)} &= 0 \end{aligned}$$

Now suppose (14) holds for $\alpha = \alpha'$.

$$\begin{aligned}
\sum_{j=r}^{\alpha'+1} (-2)^j \binom{\alpha'+1}{j} \binom{j}{r} &= \frac{\alpha'+1}{r-\alpha'-1} \sum_{j=r}^{\alpha'} (-2)^j \binom{\alpha'}{j} \binom{j}{r} && \text{by (20)} \\
&= \frac{\alpha'+1}{r-\alpha'-1} (-1)^{\alpha'} 2^r \binom{\alpha'}{r} && \text{by inductive hypothesis} \\
&= (-1)^{\alpha'+1} 2^r \frac{\alpha'+1}{\alpha'+1-r} \binom{\alpha'}{r} \\
&= (-1)^{\alpha'+1} 2^r \binom{\alpha'+1}{r}
\end{aligned}$$

Therefore (14) holds $\forall \alpha \in \mathbb{N}$ $\square$

We can thus conclude that the conditions for C-α continuity are:

$$\forall \beta \leq \alpha \,.\, Q_\beta = 2^\beta \sum_{k=0}^{\beta} (-2)^k \binom{\beta}{k} P_{n-k} \tag{21}$$

References

[1] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences.* http://www.research.att.com/∼njas/sequences/, 2001.

[2] Doron Zielberger. *EKHAD.* http://www.math.rutgers.edu/∼zeilberg/tokhniot/EKHAD.